Introduction to Ordinary Differential Equations

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Outline

- Review numerical solutions
- Basics of differential equations
- First order equations
  - Separable solutions
  - General solution for linear equation
- Introduction to second order equations
  - Problems considered
  - Basis of solutions
  - Constant-coefficient, homogenous case

Review Numerical Solutions

- Gauss elimination is basic approach
- Need pivoting strategies to reduce round-off error in solution
- Modifications of Gauss elimination
  - Gauss-Jordan sometimes used for finding inverse of matrix
  - LU method generally preferred
    - Does most of the elimination work without knowing the right-hand-side (**b**) vector
    - 1D integer vector required for pivoting

Basic Differential Equations

- A differential equation is an equation that contains derivatives of a dependent variable, e.g., y(x) or u(x,y)
- Differential equation solution gives y(x) or u(x,y) as a function of independent variable(s)
  - Ordinary differential equations (ODE) have one independent variable
  - Partial differential equations (PDE) have more than one independent variable

Definitions and Terms

- Differential equations have boundary conditions or initial conditions
- A general solution to the differential equation is one which can fit any boundary or initial condition by adjusting "constants" in the solution
- A solution that satisfies the differential equation and the boundary or initial conditions is called a particular solution

More Definitions and Terms

- The order of a differential equation is the order of the highest derivative in the equation
- A linear differential equation is one in which the dependent variable and its derivatives all appear in linear terms
- A homogenous differential equation is one in which all terms involve the dependent variable and its derivatives

Examples of ODEs

x: Independent
y: Dependent

- Third-order, linear, homogenous $\frac{d^3y}{dx^3} + \sin(x)\frac{dy}{dx} - x^2 y = 0$
- Second-order, non-linear, homogenous $\frac{d^2y}{dx^2} + \sin(y) = 0$
- Second-order, linear, non-homogenous $\frac{d^2y}{dx^2} + y = e^x \cos(x)$
- Third-order, non-linear, non-homogenous $\frac{d^3y}{dx^3} + y\frac{dy}{dx} = 1$

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Applications

- First order differential equations are often used to model rate processes
  - Newton's cooling $dT/dt = -k(T - T_\infty)$
  - chemical reactions, $dc_i/dt = f(\mathbf{c},T)$
- Newton's second law, F = ma leads to second order equations for mechanical systems $m\,d^2y_i/dt^2 = F_i$
- Deflection, y, of rectangular beam oriented in x direction $EId^4y/dx^4 = f(x)$

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Separable Forms

- Simple differential equations can be written as integrals
  - Even if numerical quadrature is required this is more accurate than numerical solution of ODE

$$\frac{dy}{dx} = f(x) \quad \Rightarrow \quad y = \int f(x)dx + C$$

$$\frac{dy}{dx} = f(x)g(y) \quad \Rightarrow \quad \int \frac{dy}{g(y)} = \int f(x)dx + C$$

$$\frac{dy}{dx} = h\left(\frac{y}{x}\right) \quad \Rightarrow \quad \int \frac{dx}{x} = \int \frac{du}{h(u) - u} + C$$

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Linear First-Order Equation

- The solution to the first-order equation $\frac{dy}{dx} + p(x)y = r(x)$
- Is given by the following result

$$y = e^{-h}\left[\int e^h r(x)dx + C\right] \quad where\ h = \int p(x)dx$$

- The constant, C, requires the specification of the value of y at a particular value of x; e.g., $y = y_1$ at x = 1

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P(x,y)dx + Q(x,y)dy = 0

- Is P(x,y)dx + Q(x,y)dy = 0 an exact form?
- From differential of a function of two variables, f(x,y), see if P and Q satisfy partial derivative relation
- If df = 0, f = C

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$P(x,y) \equiv \frac{\partial f}{\partial x} \qquad Q(x,y) \equiv \frac{\partial f}{\partial y}$$

$$df = Pdx + Qdy$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} \quad \Rightarrow \quad \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

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Exact Form

- If $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, P(x,y)dx +Q(x,y)dy = df
- We may not know (or care) what f is, but we use df = P(x,y)dx +Q(x,y)dy to solve the differential equation
- We also know that P(x,y)dx +Q(x,y)dy = 0 means that df = 0 or f = a constant
- We also know that P and Q are derivatives of this mysterious f function

$$P(x,y) = \frac{\partial f}{\partial x} \quad and \quad Q(x,y) = \frac{\partial f}{\partial y} \quad only\ if\ \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

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Exact Forms II

- Integrate df = P(x,y)dx + Q(x,y)dy for constant f (df = 0)
- f = constant, C, because df = 0

$$f = \int df = \int_{y=const} P(x,y)dx + g(y) = C$$

$$Q(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int_{y=const} P(x,y)dx\right] + \frac{dg(y)}{dy}$$

$$\frac{dg(y)}{dy} = Q(x,y) - \frac{\partial}{\partial y}\left[\int_{y=const} P(x,y)dx\right] = h(y)$$

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Exact Forms III

- Final equation must be a function of y only
- Integrate this equation for g(y)

$$\frac{dg(y)}{dy} = Q(x,y) - \frac{\partial}{\partial y}\left[\int_{y=const} P(x,y)dx\right] = h(y)$$

$$g(y) = \int dg = \int \left\{ Q(x,y) - \frac{\partial}{\partial y}\left[\int_{y=const} P(x,y)dx\right]\right\} dy + C$$

- Substitute g(y) into equation for f

$$f = \int df = \int_{y=const} P(x,y)dx + g(y) + C$$

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Exact Forms IV

$$f = \int df = \int_{y=const} P(x,y)dx + g(y) = C_1$$

$$g(y) = \int \left\{ Q(x,y) - \frac{\partial}{\partial y}\left[\int_{y=const} P(x,y)dx\right]\right\} dy + C_2$$

$$\int_{y=const} P(x,y)dx + \int \left\{ Q(x,y) - \frac{\partial}{\partial y}\left[\int_{y=const} P(x,y)dx\right]\right\} dy + C_2 = C_1$$

- Combine constants into a single constant
- Obtain implicit relationship between y and x

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Solving Exact Pdx + Qdy = 0

$$\int_{y=const} P(x,y)dx + \int \left\{ Q(x,y) - \frac{\partial}{\partial y}\left[\int_{y=const} P(x,y)dx\right]\right\} dy = C$$

- Step 1 – Integrate P(x,y)dx with y constant
- Step 2 – Take the y derivative of the step 1 result and subtract it from Q(x,y)
- Step 3 – Integrate result of step 2, that will be a function of y only, over y
- Step 4 – Add results of steps 1 and 3

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Integrating Factors

- Used to integrate P(x,y)dx + Q(x,y)dy = 0 if P and Q are not exact
- Basic idea is to find a factor, F, that multiplies the original equation: FPdx + FQdy = 0
- Find the F factor so that FP and FQ are exact
- Use trial and error or process outlined in Kreyszig to find F

$$\frac{\partial FQ}{\partial x} = \frac{\partial FP}{\partial y}$$

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First-order Equations

- First order rate equation where rate is proportional to amount dy/dt = -ky
- $y = y_0 e^{-k(t-t_0)}$
- General linear first order equation for y(x): dy/dx + f(x)y = g(x) has closed form solution shown below
- C is found from initial condition

$$p = \int f(x)dx \qquad y = e^{-p}\left[C + \int e^{p} g(x)dx\right]$$

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Existence and Uniqueness

- Important because we can try numerical solution of an ODE with no solution
- Examine dy/dx = f(x,y) with $y(x_0) = y_0$ in a region $|x – x_0| < a$ and $|y – y_0| < b$
- Derivate is bounded: $|f(x,y)| \le K$
- Equation has a solution in region $|x – x_0| < \min(a, b/K)$
- Uniqueness requires $|\partial f/\partial y| \le M$

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Existence and Uniqueness

- Example: xy' = 4, y(0) = 0
- Here we have dy/dx = f(x,y) = 4/x with $y(x_0 = 0) = y_0 = 0$
- Region is $|x – 0| < a$ and $|y – 0| < b$
- Derivative is ***not*** bounded at $x = x_0 = 0$
- Therefore have **no solutions**: $|f(x,y)| \le K$
- Attempted solution is y = 4ln(x) + C, but we cannot apply this at $x = x_0 = 0$

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Second Order Equations

- First look at homogenous linear equations $$\frac{d^2y}{dt^2} + p(x)\frac{dy}{dy} + q(x)y = 0$$
- Then consider nonhomogenous equations $$\frac{d^2y}{dt^2} + p(x)\frac{dy}{dy} + q(x)y = r(x)$$
- Most nonlinear equations require numerical solution $$\phi\left(\frac{d^2y}{dt^2}, \frac{dy}{dy}, y, x\right) = 0$$

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Linear Homogenous 2nd Order

$$\frac{d^2y}{dt^2} + p(x)\frac{dy}{dy} + q(x)y = 0$$

- Any linear combination of two solutions, y_1 and y_2, to this equation, $y = c_1y_1 + c_2y_2$, is also a solution
- Can prove this by substituting $c_1y_1 + c_2y_2$ combination into original equation for which y_1 and y_2 are solutions

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Linear Homogenous 2nd Order II

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

- A basis of solutions for this equation is any two linearly independent solutions y_1 and y_2
- $y = c_1y_1 + c_2y_2$ is a general solution where c_1 and c_2 can be used to fit initial or boundary conditions
  - Initial conditions specify y(0) and y'(0)
  - Boundary conditions specify y(a) and y(b)

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Existence and Uniqueness

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \quad \text{with } y(x_0) = K_0 \; y'(x_0) = K_1$$

- The initial value problem defined above on the open interval, I, defined as a < x < b, has a unique solution if p(x) and q(x) are continuous on the interval, I, and x_0 is located on the interval, I.
- Next slide discusses linear independence of solutions to this ODE

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Linear Independence

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \quad with\ y(x_0) = K_0\ y'(x_0) = K_1$$

- The solution to the initial value problem defined above can be written as $y(x) = k_1y_1(x) + k_2y_2(x)$ where y_1 and y_2 are linearly independent
- For $k_1y_1(x) + k_2y_2(x) = 0$, we must have $k_1 = 0$ and $k_2 = 0$ for y_1 and y_2 to be linearly independent

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Wronski Determinant

- $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$
- W is called the Wronski determinant or Wronskian for the ODE under discussion

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \quad with\ y(x_0) = K_0\ y'(x_0) = K_1$$

- The solutions to the ODE are linearly independent if there is some point, x_1, in the solution interval for which W is not 0

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Constant Coefficients

- Simplest solution is when p(x) and q(x) are constants
- The differential equation for this case is shown below
- The solution is shown on the next slide

$$c\frac{d^2y}{dx^2} + d\frac{dy}{dx} + ey = 0 \quad \Rightarrow \quad \frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = 0$$

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Constant Coefficient Solution

- ODE solutions are based on solution of characteristic equation $\lambda^2 + a\lambda + b = 0$
- Solutions for λ are roots of quadratic

$$\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = 0 \qquad \lambda_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \qquad \lambda_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}$$

- Two linearly-independent ODE solutions are $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$

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Constant Coefficient Solution

- Two solutions, $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$, form basis for general solution
- General solution is $y = C_1y_1 + C_2y_2$

$$\lambda_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \qquad \lambda_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}$$

- Can show solution is correct by substituting into original ODE
- Find C_1 and C_2 from initial or boundary conditions

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Constant Coefficient Solution

- General solution is linear combination of linearly-independent solutions
- $y = C_1y_1 + C_2y_2 = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$
- Derivatives of the individual solutions are $y_1' = \lambda_1 e^{\lambda_1 x}$ and $y_2' = -\lambda_2 e^{\lambda_2 x}$; $y_1'' = \lambda_1^2 e^{\lambda_2 x}$ and $y_2'' = \lambda_2^2 e^{\lambda_2 x}$
- Use these derivatives (plus $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$) to verify individual solutions

$$Does\ \frac{d^2y_i}{dt^2} + \alpha\frac{dy_i}{dy} + \beta y_i = \lambda_i^2 e^{\lambda_i x} + \alpha\lambda_i e^{\lambda_i x} + \beta e^{\lambda_i x} = 0?$$

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Constant Coefficient Solution

$$\lambda_i^2 e^{\lambda_i x} + \alpha\lambda_i e^{\lambda_i x} + \beta e^{\lambda_i x} = 0 \quad \Rightarrow \quad \boxed{\lambda_i^2 + \alpha\lambda_i + \beta = 0}$$

Characteristic equation

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} = \frac{-\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}$$

Definition: $\mp = -(\pm)$

$$\lambda^2 = \left(\frac{-\alpha}{2}\right)^2 \mp 2\frac{\alpha}{2}\sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta} + \left(\sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}\right)^2 = 2\left(\frac{\alpha}{2}\right)^2 - \beta \mp \alpha\sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}$$

$$\lambda^2 + \alpha\lambda + \beta = 2\left(\frac{\alpha}{2}\right)^2 - \beta \mp \alpha\sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta} + \alpha\left[\frac{-\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}\right] + \beta = 0$$

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Exercise

- Find the solution to the following ODE with initial conditions that y(0) = 1 and y'(0) = 0 $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + \frac{5y}{4} = 0$$
- Repeat the solution to the ODE with boundary conditions that y(0) = 1 and y(1) = 0

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Exercise Solution (1/3)

- ODE has form discussed previously with α = 3 and β = 5/4 $$\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = 0$$
- Solution is $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$ where

$$\lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} = \frac{-3 \pm \sqrt{3^2 - 4(5/4)}}{2} = \frac{-3 \pm \sqrt{4}}{2} = \frac{-1}{2}, \frac{-5}{2}$$

$$\boxed{y = C_1 e^{-x/2} + C_2 e^{-5x/2}}$$

Apply boundary conditions to find C_1 and C_2

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Exercise Solution (2/3)

- First case y(0) = 1, y'(0) = 0

$$y(0) = 1 = C_1 e^{-0/2} + C_2 e^{-5(0)/2} = C_1 + C_2$$

$$y'(0) = 0 = -\frac{1}{2}C_1 e^{-0/2} - \frac{5}{2}C_2 e^{-5(0)/2} = -\frac{C_1}{2} - \frac{5C_2}{2} \Rightarrow C_1 = -5C_2$$

$$C_1 + C_2 = 1 = -5C_2 + C_2 = -4C_2 \Rightarrow C_2 = -1/4 \qquad C_1 = -5C_2 = 5/4$$

$$y = C_1 e^{-x/2} + C_2 e^{-5x/2} \qquad \boxed{y = \frac{5}{4}e^{-x/2} - \frac{1}{4}e^{-5x/2}}$$

- We can show that y(0) = 1 and y'(0) = 0

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Exercise Solution (3/3)

- Second case y(0) = 1, y(1) = 0

$$y(0) = 1 = C_1 e^{-0/2} + C_2 e^{-5(0)/2} = C_1 + C_2 \Rightarrow C_1 = 1 - C_2$$

$$y(1) = 0 = C_1 e^{-1/2} + C_2 e^{-5(1)/2} = (1 - C_2)e^{-1/2} + C_2 e^{-5(1)/2}$$

$$C_2\left(e^{-1/2} - e^{-5(1)/2}\right) = e^{-1/2} \qquad C_2 = \frac{e^{-1/2}}{e^{-1/2} - e^{-5/2}} = \frac{1}{1 - e^{-2}}$$

$$C_1 = 1 - C_2 = 1 - \frac{1}{1 - e^{-2}} = \frac{1 - e^{-2} - 1}{1 - e^{-2}} = \frac{-e^{-2}}{1 - e^{-2}}$$

$$y = C_1 e^{-x/2} + C_2 e^{-5x/2} \qquad y = \frac{-e^{-2}}{1 - e^{-2}}e^{-x/2} + \frac{1}{1 - e^{-2}}e^{-5x/2}$$

- We can show that y(0) = 1 and y(1) = 0

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Check General Solution (1/2)

- Plug solution into original ODE

$$y = C_1 e^{-x/2} + C_2 e^{-5x/2}$$

$$y' = -\frac{1}{2}C_1 e^{-x/2} - \frac{5}{2}C_2 e^{-5x/2}$$

$$y'' = \frac{1}{4}C_1 e^{-x/2} + \frac{25}{4}C_2 e^{-5x/2}$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + \frac{5}{4}y = \frac{1}{4}C_1 e^{-x/2} + \frac{25}{4}C_2 e^{-5x/2}$$

$$+3\left[-\frac{1}{2}C_1 e^{-x/2} - \frac{5}{2}C_2 e^{-5x/2}\right] + \frac{5}{4}\left[C_1 e^{-x/2} + C_2 e^{-5x/2}\right]$$

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Check General Solution (2/2)

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+\frac{5}{4}y=\frac{1}{4}C_1e^{-x/2}+\frac{25}{4}C_2e^{-5x/2}$$

$$+3\left[-\frac{1}{2}C_1e^{-x/2}-\frac{5}{2}C_2e^{-5x/2}\right]+\frac{5}{4}\left[C_1e^{-x/2}+C_2e^{-5x/2}\right]$$

$$=\left[\frac{1}{4}-\frac{3}{2}+\frac{5}{4}\right]C_1e^{-x/2}+\left[\frac{25}{4}-\frac{15}{2}+\frac{5}{4}\right]C_2e^{-5x/2}$$

$$=\left[\frac{1-6+5}{4}\right]C_1e^{-x/2}+\left[\frac{25-30+5}{4}\right]C_2e^{-5x/2}=0$$

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